[T.K] 2.6.5.2.

Define $\lfloor\ \rfloor : \mathbb{R}_+ \to \mathbb{N}$
$x \mapsto \lfloor x \rfloor :=$ the largest integer smaller than x (e.g $\lfloor 3.1 \rfloor = \lfloor 3.9 \rfloor = \lfloor 3 \rfloor = 3$)

Let $X \in Exp(\frac{1}{\lambda})$, $\lambda > 0$ and set $L_m := \lfloor \frac{X}{m} \rfloor$ and $R_m = X - m \cdot L_m$; $m \in \mathbb{N}$.

To show : * L_m and R_m are independent
* Determine their marginal distributions.

Remarks : * L_m is the result of the euclidean division of X by m and R_m is the rest of this division.
* Be careful : X and R_m have continuous p.d.f. but L_m is a discrete random variable !
i.e $L_m \in \mathbb{N}$.

Reminders : Let Z_1, Z_2 be two random variables, Z_1 and Z_2 are independent iff $\forall A, B$ in the σ-algebra,
$\mathbb{P}(Z_1 \in A, Z_2 \in B) = \mathbb{P}(Z_1 \in A) \cdot \mathbb{P}(Z_2 \in B)$

Equivalently, $\mathbb{P}(Z_1 \leq t_1, Z_2 \leq t_2) = \mathbb{P}(Z_1 \leq t_1)\mathbb{P}(Z_2 \leq t_2)$ $\forall t_1, t_2 \in \mathbb{R}$

If (Z_1, Z_2) has a continuous p.d.f, it is equivalent to $f_{(Z_1,Z_2)}(u,v) = f_{Z_1}(u) f_{Z_2}(v)$ $\forall u, v \in \mathbb{R}$.

Now if Z_1 is discrete, Z_1 and Z_2 are independent iff $\forall z \in \mathbb{N}, \forall t_2 \in \mathbb{R}$, $\mathbb{P}(Z_1 = z, Z_2 \leq t_2) = \mathbb{P}(Z_1 = z)\mathbb{P}(Z_2 \leq t_2)$.

Finally, $X \in Exp(\frac{1}{\lambda})$, $\lambda > 0$ iff $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \forall x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

As in session 2, if $\mathbb{P}(Z_1 \leq t_1, Z_2 \leq t_2) = g_1(t_1) \cdot g_2(t_2)$, then we can conclude that Z_1 and Z_2 are independent. Indeed, up to multiplication by a constant, $g_1 = F_{Z_1}$ and $g_2 = F_{Z_2}$.

Sketch of the proof : 1./ Compute $\mathbb{P}(L_m = n, R_m \leq r)$. Idea : reexpress $\{L_m = n, R_m \leq r\}$ as an event depending on X

2./ Check that the expression that we get factorises.

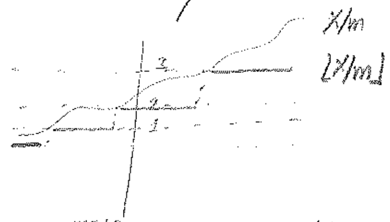
3./ Compute $\mathbb{P}(L_m = n)$ and $\mathbb{P}(R_m \leq r)$.

Take $n \in \mathbb{N}$ and $r \le m$.

1./ $\{L_m = n,\ R_m \le r\} = \{\lfloor \frac{X}{m} \rfloor = n,\ X - \lfloor \frac{X}{m} \rfloor \cdot m \le r\} = \{\frac{X}{m} \in [n, n+1),\ X - m \cdot n \le r\}$

$= \{X \in [mn, mn+r]\}$

$X \le mn + r$



$$\Rightarrow \mathbb{P}(L_m = n, R_m \le r) = \mathbb{P}(X \in [mn, mn+r]) = \int_{mn}^{mn+r} f_X(t)\,dt = \int_{mn}^{mn+r} \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_{mn}^{mn+r} = e^{-\lambda mn} - e^{-\lambda(mn+r)}$$

2./

$$= \underbrace{e^{-\lambda mn}}_{g_1} \underbrace{(1 - e^{-\lambda r})}_{g_2}$$ $\Rightarrow$ We can already conclude that L_m and R_m are independent.

3./ $$\mathbb{P}(L_m = n) = \mathbb{P}(\lfloor \tfrac{X}{m} \rfloor = n) = \mathbb{P}(\tfrac{X}{m} \in [n, n+1)) = \mathbb{P}(X \in [mn, mn+m)) = \int_{mn}^{mn+m} f_X(t)\,dt = \int_{mn}^{mn+m} \lambda e^{-\lambda t}\,dt$$

$$= e^{-\lambda mn}(1 - e^{-\lambda m})$$

and $$\mathbb{P}(R_m \le r) = \sum_{n \ge 0} \mathbb{P}(L_m = n, R_m \le r) = \sum_{n \ge 0} e^{-\lambda mn}(1 - e^{-\lambda r}) = (1 - e^{-\lambda r}) \underbrace{\sum_{n \ge 0} (e^{-\lambda m})^n} = \frac{1 - e^{-\lambda r}}{1 - e^{-\lambda m}}$$

geometric serie: $\frac{1}{1 - e^{-\lambda m}}$

$\sum_{k \ge 0} x^k = \frac{1}{1-x} \quad \forall x \in (-1, 1)$

QED

[T.K] 3.8.8.5

Let $X_1, X_2, \dots, X_n$ be independent and Poisson (λ) - distributed random variables, respectively.

Let $I \in U(1, 2, \dots, n)$ be a random variable independent from the X_i's

To determine : $\mathbb{E}[X_I]$ and $Var[X_I]$

Reminder : $X_i \in$ Poisson (λ) iif $\mathbb{P}(X_i = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \forall k \in \mathbb{N}$.

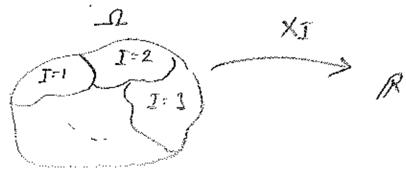
$I \in U(1, 2, \dots, n)$ iif $\mathbb{P}(I = i) = \frac{1}{n} \quad \forall i \in \{1, 2, \dots, n\}$.

$X_I : \Omega \to \mathbb{R}$

$$\mathbb{E}[X_I] = \sum_{i=1}^{n} \mathbb{E}[X_I \cdot 1\!\!1_{\{I = i\}}] = \sum_{i=1}^{n} \mathbb{E}[X_I \mid I = i] \cdot \mathbb{P}(I = i)$$

$X_I = \sum_{i=1}^{n} X_I\, 1\!\!1_{\{I=i\}}$, where $1\!\!1_{\{I=i\}} = \begin{cases} 1 & \text{if } I = i \\ 0 & \text{otherwise} \end{cases}$

indeed $\sum_{i=1}^{n} 1\!\!1_{\{I=i\}} = 1$



and $X_I = X_i\, 1\!\!1_{\{I=i\}}$ on $\{I = i\}$

$\mathbb{E}[1\!\!1_G] = \mathbb{P}(G) \quad \forall$ event G.

$$= \int 1\!\!1_G\, d\mathbb{P} = \int_G d\mathbb{P} = \mathbb{P}(G)$$

$$(*) \overset{ind}{=} \sum_{i=1}^{n} \mathbb{E}[X_i]\, \mathbb{P}(I=i) = \sum_{i=1}^{n} \mathbb{E}[X_i] \cdot \frac{1}{n}$$

$$\cdot\ \mathbb{E}[X_i] = \sum_{K\geq 0} K \cdot \frac{\lambda_i^K}{K!} e^{-\lambda_i}$$ Remmder : $e^{\lambda_i} = \sum_{K\geq 0} \lambda_i^K$

$$= \sum_{K\geq 1} \frac{\lambda_i^K}{(K-1)!} e^{-\lambda_i}$$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \dots$ (Taylor expansion)

Take $f(x) = e^{\lambda_i x}$, $f^{(K)}(x) = \lambda_i^K e^{\lambda_i x}$

$$= e^{-\lambda_i}\lambda_i \cdot \sum_{K\geq 1} \frac{\lambda_i^{K-1}}{(K-1)!}$$

Then $f(x) = \sum_{K\geq 0} \frac{\lambda_i^K \cdot x}{K!} \Rightarrow f(1) = \sum_{K\geq 0} \frac{\lambda_i^K}{K!}$ and $f(1) = e^{\lambda_i}$

$$= e^{-\lambda_i}\lambda_i \sum_{K\geq 0} \frac{\lambda_i^K}{K!} = \lambda_i \qquad \Rightarrow \sum_{i=1}^{n} \mathbb{E}[X_i]\cdot\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\lambda_i$$

Now $Var(X_I) = \mathbb{E}[X_I^2] - \mathbb{E}[X_I]^2$

$$\mathbb{E}[X_I^2] = \sum_{i=1}^{n} \mathbb{E}[X_I^2 \mathbb{1}_{\{I=i\}}] \underset{/}{=} \sum_{i=1}^{n} \mathbb{E}[X_i^2]\,\mathbb{P}(I=i) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i^2]$$

Remark : Z_1 and Z_2 independent
$\Rightarrow f(Z_1)$ and Z_2 independent $\forall f: \mathbb{R}\to\mathbb{R}$.

$$\mathbb{E}[X_i^2] = \sum_{K\geq 0} K^2\, \mathbb{P}(X_i^2 = K^2) = \sum_{K\geq 0} K^2\, \mathbb{P}(X_i = K) = \sum_{K\geq 0} K^2 \frac{\lambda_i^K}{K!} e^{-\lambda_i} = e^{-\lambda_i}\sum_{K\geq 0} K^2 \frac{\lambda_i^K}{K!}$$

$$\text{Now } \frac{d}{d\lambda_i} \underbrace{\sum_{K\geq 0} K \frac{\lambda_i^K}{K!}}_{} = \sum_{K\geq 0}\frac{K}{K!}\frac{d}{d\lambda_i}\lambda_i^K = \sum_{K\geq 0}\frac{K}{K!} K\cdot\lambda_i^{K-1} = \sum_{K\geq 0}\frac{K^2\lambda_i^{K-1}}{K!} = \frac{1}{\lambda_i}\sum_{K\geq 0}\frac{K^2\lambda_i^K}{K!}$$

as before

$$= \lambda_i \underbrace{\sum_{K\geq 0}\frac{\lambda_i^K}{K!}}_{e^{\lambda_i}} \qquad \text{and} \quad \frac{d}{d\lambda_i}\lambda_i e^{\lambda_i} = e^{\lambda_i} + \lambda_i e^{\lambda_i}$$

$$\Rightarrow \sum_{K\geq 0}\frac{K^2\lambda_i^K}{K!} = \lambda_i\left(e^{\lambda_i} + \lambda_i e^{\lambda_i}\right)$$

$$\Rightarrow \mathbb{E}[X_i^2] = e^{-\lambda_i}\sum_{K\geq 0} K^2\frac{\lambda_i^K}{K!} = \lambda_i(1+\lambda_i) = \lambda_i^2 + \lambda_i$$

$$\Rightarrow \mathbb{E}[X_I^2] = \frac{1}{n}\sum_{i=1}^{n}(\lambda_i^2 + \lambda_i)$$

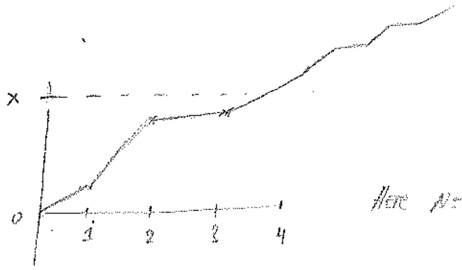
$$\text{Now } \mathbb{E}[X_I]^2 = \left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right)^2 \Rightarrow Var[X_I] = \frac{1}{n}\sum_{i=1}^{n}\lambda_i^2 + \frac{1}{n}\sum_{i=1}^{n}\lambda_i - \left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right)^2$$

[T.K] 3.8.3.6 Let $X_1, X_2, \dots, X_n$ be independent and identically $Exp(\frac{1}{\lambda})$ - distributed random variables

Set $S_n := X_1 + \dots + X_n \quad \forall n \geq 1$ and $N := \max\{n ; S_n \leq x\}$, where $x \in \mathbb{R}_+$
$S_0 = 0$

To show : $N \in Poisson(\lambda x)$

Heuristics :



N is the last time such that $S_N \leq x$.

Here $N = 3$.

Fix $n \geq 0$.

$$\{N=n\} = \{S_n \leq x \text{ and } S_{n+1} > x\} = \left\{\sum_{i=1}^{n} X_i \leq x \text{ and } \sum_{i=1}^{n+1} X_i > x\right\} = \left\{\sum_{i=1}^{n} X_i \leq x \text{ and } X_{n+1} > x - \sum_{i=1}^{n} X_i\right\}$$
$$= \{S_n \leq x \text{ and } X_{n+1} > x - S_n\}$$

$$\Rightarrow \mathbb{P}(N=n) = \mathbb{P}(S_n \leq x \text{ and } X_{n+1} > x - S_n)$$

$$= \int_{-\infty}^{x} \int_{x-u}^{+\infty} f_{(S_n, X_{n+1})}(u,v)\, dv\, du \overset{ind.}{=} \int_{-\infty}^{x} \int_{x-u}^{+\infty} f_{S_n}(u) f_{X_{n+1}}(v)\, dv\, du$$

We need to compute f_{S_n}.

$$\mathbb{P}(S_n \leq u) = \mathbb{P}(X_1 + X_2 + \dots + X_n \leq u) = \mathbb{P}(X_1 + \dots + X_{n-1} \leq u - X_n,\ X_n \in \mathbb{R}) = \mathbb{P}(S_{n-1} \leq u - X_n,\ X_n \in \mathbb{R}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{u-v} f_{S_{n-1}}(t) f_{X_n}(v)\, dt\, dv$$

$$\Rightarrow f_{S_n}(u) = \frac{d}{du}\mathbb{P}(S_n \leq u) = \frac{d}{du} \int_{-\infty}^{+\infty} \int_{-\infty}^{u-v} f_{S_{n-1}}(t) f_{X_n}(v)\, dt\, dv = \int_{-\infty}^{+\infty} f_{S_{n-1}}(u-v) f_{X_n}(v)\, dv$$

$\Rightarrow$ We have a nice recurrence relation! Let's determine f_{S_n} by induction!

$$f_{S_2}(u) = \int_{-\infty}^{+\infty} f_{S_1}(u-v) f_{X_2}(v)\, dv = \int_{-\infty}^{+\infty} f_{X_1}(u-v) f_{X_2}(v)\, dv = \int_{-\infty}^{+\infty} \lambda e^{-\lambda(u-v)} \mathbb{1}_{\{u-v \geq 0\}} \lambda e^{-\lambda v} \mathbb{1}_{\{v \geq 0\}}\, dv$$

$$= \int_0^u \lambda^2 e^{-\lambda u}\, dv = \lambda^2 e^{-\lambda u} \cdot u$$

$$f_{S_3}(u) = \int_{-\infty}^{+\infty} f_{S_2}(u-v) f_{X_3}(v)\, dv = \int_{-\infty}^{+\infty} \lambda^2 e^{-\lambda(u-v)} (u-v) \mathbb{1}_{\{u-v \geq 0\}} \lambda e^{-\lambda v} \mathbb{1}_{\{v \geq 0\}}\, dv = \lambda^3 e^{-\lambda u} \underbrace{\int_0^u (u-v)\, dv}_{-\frac{1}{2}(u-v)^2 \big|_0^u} = \lambda^3 e^{-\lambda u} \cdot \frac{1}{2} u^2$$

Suppose $f_{S_k}(u) = \lambda^k e^{-\lambda u} \cdot \frac{1}{(k-1)!} u^{k-1}$, $u \geq 0$, $k \geq 1$. Then

$$f_{S_{k+1}}(u) = \int_{-\infty}^{+\infty} f_{S_k}(u-v) f_{X_{k+1}}(v)\, dv = \int_{-\infty}^{+\infty} \lambda^k e^{-\lambda(u-v)} \frac{1}{(k-1)!} (u-v)^{k-1} \lambda e^{-\lambda v} \mathbb{1}_{\{u-v \geq 0\}} \mathbb{1}_{\{v \geq 0\}}\, dv$$

$$= \int_0^u \lambda^{k+1} e^{-\lambda u} \frac{1}{(k-1)!} (u-v)^{k-1}\, dv = \lambda^{k+1} e^{-\lambda u} \cdot \frac{1}{(k-1)!} \left[-\frac{1}{k}(u-v)^k\right]\Big|_0^u = \lambda^k e^{-\lambda u} \cdot \frac{1}{k!} u^k \quad \Rightarrow \text{OK!}$$

$$\text{Hence } \mathbb{P}(N=n) = \int_{-\infty}^{x} \int_{x-u}^{+\infty} \lambda^n e^{-\lambda u} \frac{1}{(n-1)!} u^{n-1} \mathbb{1}_{\{u \geq 0\}} \lambda e^{-\lambda v} \mathbb{1}_{\{v \geq 0\}}\, dv\, du = \frac{\lambda^{n+1}}{(n-1)!} \int_0^x u^{n-1} e^{-\lambda u} \underbrace{\int_{x-u}^{+\infty} e^{-\lambda v}\, dv}_{-\frac{1}{\lambda} e^{-\lambda v} \big|_{x-u}^{+\infty} = \frac{1}{\lambda} e^{-\lambda(x-u)}}\, du = \frac{\lambda^n}{(n-1)!} e^{-\lambda x} \int_0^x u^{n-1}\, du = \frac{\lambda^n}{n!} e^{-\lambda x} \cdot x^n$$

[T.K] 3.8.3.10 : Let $X \in Exp(\frac{1}{\lambda})$ and $Y \in Exp(\frac{1}{\lambda})$, $\lambda > 0$ be two independent random variables.

To show : $X | X+Y = z \in U(0,z)$

By definition, $$f_{X|X+Y=z}(u) = \frac{f_{(X,X+Y)}(u,z)}{f_{X+Y}(z)}$$

$$\mathbb{P}((X, X+Y) \in (-\infty,\tilde{x}] \times (-\infty,\tilde{y}]) = \mathbb{P}((X,Y) \in h((-\infty,\tilde{x}] \times (-\infty,\tilde{y}])) = \iint_{h((-\infty,\tilde{x}]\times(-\infty,\tilde{y}])} f_X(x) f_Y(y)\, dy\, dx = \int_{-\infty}^{\tilde{x}}\int_{-\infty}^{\tilde{y}} f_X(u) f_Y(z-u)\, 1\, dz\, du$$

we want h ; $h(x, x+y) = (x,y)$

Set $u = x$, $z = x+y$ $\Rightarrow \begin{cases} u = x \\ z-u = y \end{cases} \Rightarrow h(u,z) = (u, z-u) \Rightarrow J_h = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $|J_h| = 1$.

$$= \int_{-\infty}^{\tilde{x}}\int_{-\infty}^{\tilde{y}} \lambda e^{-\lambda u} \lambda e^{-\lambda(z-u)} \mathbb{1}_{\{u \geq 0\}}(u) \mathbb{1}_{\{z-u \geq 0\}}(z)\, dz\, du = \int_{-\infty}^{\tilde{x}}\int_{-\infty}^{\tilde{y}} \lambda^2 e^{-\lambda z} \mathbb{1}_{\{u \geq 0\}}(u) \mathbb{1}_{\{z \geq u\}}(z)\, dz\, du$$

$$\Rightarrow f_{(X,X+Y)}(u,z) = \lambda^2 e^{-\lambda z} \mathbb{1}_{\{u \geq 0\}}(u) \mathbb{1}_{\{z \geq u\}}(z)$$

Now $$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y)\, dy = \int_{-\infty}^{+\infty} \lambda e^{-\lambda(z-y)} \mathbb{1}_{\{z-y \geq 0\}}(y)\, \lambda e^{-\lambda y} \mathbb{1}_{\{y \geq 0\}}(y)\, dy = \int_0^z \lambda^2 e^{-\lambda z}\, dy = \lambda^2 e^{-\lambda z} \cdot z\, \mathbb{1}_{\{z \geq 0\}}$$

$$\Rightarrow f_{X|X+Y=z}(u) = \frac{1}{z} \mathbb{1}_{\{u \leq z\}}(u)$$: the uniform distribution $U(0,z)$